Analysis of the stability of axisymmetric jets

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This paper is a contribution to the mathematical analysis of the stability of steady axisymmetric parallel flows of uniform fluid in the absence of rigid boundaries. A jet at sufficiently high Reynolds number for the angle of viscous spreading to be small is the typical example of the primary flows considered, and the theoretical velocity profile far downstream in such a jet is kept in mind continually. It is obvious from experience that such jets are unstable, presumably to infinitesimal disturbances, but there is little observational data about the critical Reynolds number or the mode of disturbance that grows most rapidly at a given Reynolds number.

The typical small disturbance considered is a Fourier component with sinusoidal dependence on both αx and $n\phi$ (x, r, ϕ are cylindrical polar co-ordinates). There is no analogue of Squire's theorem for two-dimensional primary flows, and both α and n are essential parameters of the disturbance. We have concentrated on the stability characteristics in the limit of large Reynolds number, and have aimed in particular at determining the (integral) value of n at which the growth rate is a maximum in these simpler circumstances.

A number of general results for inviscid fluid are established, many of them analogues of corresponding results for two-dimensional primary flow. A necessary condition for the existence of amplified disturbances is that

$$Q(r) = rU'/(n^2+\alpha^2r^2)$$

should have a numerical maximum at some point in the fluid; this condition is satisfied for all n in the case of a cylindrical shear layer or 'top-hat' jet profile (for which a complete solution of the disturbance equation can be obtained), and for $n \geqslant 1$ in the case of a 'far-downstream' jet profile. The wave speed c_r of a neutral disturbance is equal to the value of U either at the point where $dQ/dr = 0$ or at $r = 0$. In the latter case the eigen-function (if one exists) is singular at the axis in general; the former case is presumably relevant to the 'upper branch' of the curve of neutral stability (for given n). The Reynolds stress due to the disturbance acting across a cylindrical surface is examined. Here, as in some other contexts, it is useful to consider components of velocity parallel and perpendicular to a circular helix on which the phase of the disturbance wave is constant. For a neutral disturbance the component of disturbance velocity parallel to the local wave helix is infinite at the critical point where $U = c_r$ (corresponding to the known singularity for a three-dimensional disturbance to two-dimensional flow), and there is a peak in the Reynolds stress there.

It is shown from the form of the disturbance equation that there is an upper limit to the value of n ($\neq 0$) for a neutral (inviscid) disturbance with c_r equal to

the value of U at the point where $Q' = 0$. In the case of a jet with a 'far-downstream' profile, only the value $n = 1$ satisfies this restriction; thus only the sinuous mode $n = 1$ can yield amplified disturbances in an inviscid fluid. A numerical investigation shows that for this profile the wave-number of the neutral disturbance with $n = 1$ is $\alpha = 1{\cdot}46$.

Contents

1. Introduction

We are concerned in this paper with the stability of steady unbounded axisymmetric flows of the wake-jet type, without azimuthal swirl and with nearly parallel streamlines. The fluid is uniform throughout; that is, the jets are 'submerged'. Flows of this kind may be produced readily in a laboratory, and occur in many branches of engineering. Nevertheless, very little is known about their stability, apart from the simple fact that wakes and jets can be observed in both laminar and turbulent form and are evidently unstable under certain conditions. Two-dimensional flows have been given most of the attention in stability theory, and the limited amount of work on axisymmetric flows has been confined almost wholly to the effect of axisymmetric disturbances. The classical axisymmetric case of Poiseuille flow in a circular tube still awaits a solution despite having been virtually the first problem of hydrodynamic stability to be posed.

Steady axisymmetric wakes and jets change in width with distance downstream under the influence of viscosity. Far enough downstream from the source, the axial component of velocity (relative to that far from the axis of symmetry) takes an asymptotic form

$$x^{-p}f(r/x^q),$$

where x, r, ϕ are cylindrical co-ordinates. The values of p and q required for compatibility of this form with the boundary-layer forms of the equations of motion and continuity are as follows:

	p	q	$q-p$
Jet	1	1	0
Wake	1	$\frac{1}{2}$	$-\frac{1}{2}$

The local Reynolds number formed from the width and maximum relative velocity varies as x^{q-p}; it is constant for a jet, as is evident directly from the requirement of constancy of the flux of axial momentum across planes normal to the axis, and decreases downstream for a wake. (In two dimensions the local Reynolds number for a jet increases as $x^{\frac{1}{3}}$ and is constant for a wake.) A steady wake is therefore certain to be stable far enough downstream, and likely to be most unstable, if anywhere, in the neighbourhood of the source of the wake where the profile is not yet in its asymptotic form. This fact limits the value of theoretical analysis for wakes, and in the remainder of the paper we shall refer only to jets.

The exact solution of the Navier–Stokes equation for the stream function Ψ representing a steady jet due to a force applied at a point in an unbounded fluid (see Landau & Lifschitz 1959, p. 86) is

$$\Psi = \frac{2\nu r \sin\theta}{\sec\theta_0 - \cos\theta}, \tag{1.1}$$

where θ $(= \tan^{-1} r/x)$ is the spherical polar angle with origin at the point of application of the force and $\theta = 0$ in the direction of the force. The constant θ_0 is the value of θ at which the streamlines are at their minimum distance from the axis of symmetry and is thus a measure of the angular width of the jet. θ_0 is uniquely related to the force M applied to the fluid by

$$\frac{M}{2\pi\rho\nu^2} = \frac{32}{3}\frac{\cos\theta_0}{\sin^2\theta_0} + \frac{4}{\cos^2\theta_0}\log\left(\frac{1-\cos\theta_0}{1+\cos\theta_0}\right) + \frac{8}{\cos\theta_0}, \tag{1.2}$$

which reduces to

$$\frac{M}{2\pi\rho\nu^2} \sim \tfrac{32}{2}\theta_0^{-2} \tag{1.3}$$

for large values of M and small values of θ_0. In these latter circumstances, M is approximately equal to the flux of momentum across any plane normal to the axis and the stream function (1.1) becomes

$$\Psi \sim \frac{4\nu x r^2}{r^2 + r_0^2} \tag{1.4}$$

for $\theta \leqslant \theta_0 \ll 1$, where $r_0 = x\tan\theta_0$ is the semi-width of the jet at distance x from the source. The axial component of velocity corresponding to (1.4) is

$$\begin{aligned} U &= \frac{1}{r}\frac{\partial\Psi}{\partial r} \\ &= \frac{8\nu x r_0^2}{(r^2+r_0^2)^2} = \frac{U_0}{(1+r^2/r_0^2)^2}, \end{aligned} \tag{1.5}$$

where U_0 is the maximum value of U on the plane x = const. The relation (1.5) is familiar as the solution of the boundary-layer form of the governing equations. The Reynolds number of the jet formed from the semi-width r_0 and the maximum velocity U_0 is then

$$R = \frac{r_0 U_0}{\nu} = \frac{8}{\theta_0} = \left(\frac{3M}{\pi\rho\nu^2}\right)^{\frac{1}{2}}. \tag{1.6}$$

In the case of an experimental jet produced by the discharge of fluid from an orifice, evaluation of the corresponding value of M requires a knowledge of the distribution of velocity across the orifice. It is likely that only the volume flux Q and the orifice area A will be known with any precision, and it is therefore worth noting the following two limiting cases:

(*a*) for uniform velocity at the orifice

$$M = \frac{\rho Q^2}{A}, \quad R = \left(\frac{3}{\pi A}\right)^{\frac{1}{2}} \frac{Q}{\nu}; \tag{1.7}$$

(*b*) for a parabolic distribution of velocity at a circular orifice,

$$M = \frac{4\rho Q^2}{3A}, \quad R = \left(\frac{4}{\pi A}\right)^{\frac{1}{2}} \frac{Q}{\nu}. \tag{1.8}$$

It is a matter of convenience in this latter case that R is identical with the Reynolds number formed from the diameter of the orifice $(4A/\pi)^{\frac{1}{2}}$ and the mean velocity Q/A at the orifice.

All these formulae from (1.2) onwards, and the equating of M to the flux of momentum across a plane normal to the axis, are accurate only when $\theta_0 \ll 1$, or, equivalently, when $R/8 \gg 1$. In the subsequent stability analysis we shall be obliged to make the conventional assumption that the flow is locally unidirectional. For this to be valid it is necessary that the quantities r_0 and U_0 in (1.5) remain nearly constant over one wavelength of the disturbance. Now

$$r_0 = x \tan\theta_0 \approx \frac{8x}{R}, \tag{1.9}$$

and

$$U_0 = \frac{8\nu x}{r_0^2} \approx \frac{\nu R^2}{8x}, \tag{1.10}$$

so that both quantities are approximately constant over a wavelength if $\alpha x \gg 1$, where α is the wave-number of the disturbance. This latter condition can also be written as

$$\tfrac{1}{8}\alpha r_0 R \gg 1. \tag{1.11}$$

Surprising though it may be, there appears not to be any published data concerning the critical Reynolds number at which an axisymmetric jet becomes unstable. The only observations known to us are two of a preliminary nature not yet in print. In 1958, H. Schade at Cambridge observed that steady liquid-into-liquid jets with Reynolds numbers of several hundreds could be obtained, although no definite critical Reynolds number was found.† In 1960, A. Viilu at Massachusetts Institute of Technology observed that a liquid-into-liquid jet

† These observations were undertaken as a part of a general, and primarily theoretical, investigation by Schade of the stability of axisymmetric flows, partly at Cambridge and partly at Berlin. The mathematical work, which has some points in common with the present paper, is described in a thesis submitted in 1960 for a doctoral degree at Technische Universität, Berlin.

became unsteady at a Reynolds number of about 11.† The marked difference between these two sets of data led Dr A. J. Reynolds to undertake the experiments described in the accompanying paper (Reynolds 1962). These experiments make it clear that there is scope for the use of the assumption, so convenient in stability analysis, that the Reynolds number of the flow is large and the flow is unidirectional. They also show that small-disturbance theory is unlikely to be able to account completely for the behaviour of a real jet.

2. Equations for a small disturbance to a unidirectional jet

We suppose here that a steady jet with velocity $\mathbf{U}$, having components $U(r)$, 0, 0 relative to cylindrical co-ordinates x, r, ϕ, is subjected to a small disturbance. The velocity perturbation is $\mathbf{u}$, with components u_x, u_r, u_ϕ. Then, to the first order in $|\mathbf{u}|$, the equation of motion gives

$$\frac{\partial \mathbf{u}}{\partial t}+U\frac{\partial \mathbf{u}}{\partial x}+v\frac{d\mathbf{U}}{dr} = -\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{u}, \tag{2.1}$$

where p is the pressure perturbation. From the equation of continuity we have

$$\nabla\,.\,\mathbf{u} = \frac{\partial u_x}{\partial x}+\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\phi}{\partial \phi} = 0. \tag{2.2}$$

Without restricting the form of the disturbance we may resolve it into Fourier components with respect to the azimuthal angle ϕ; for a typical component, whose behaviour is independent of that of other components,

$$u_x, u_r, u_\phi \propto e^{in\phi}.$$

It is also possible to make a Fourier resolution with respect to x for a disturbance of finite total energy. Finally, we look for a normal-mode type of disturbance which retains the same form while its magnitude varies exponentially in t. The type of disturbance to be considered is then

$$\left.\begin{aligned} u_x, u_r, u_\phi &= \mathscr{R}[\{F(r),\, iG(r),\, H(r)\}\,e^{in\phi+i\alpha(x-ct)}],\\ p/\rho &= \mathscr{R}[P(r)\,e^{in\phi+i\alpha(x-ct)}],\end{aligned}\right\} \tag{2.3}$$

where α is a wave-number and the imaginary part of the complex velocity $c = c_r+ic_i$ determines the stability of the jet to this particular disturbance. We have put $u_r \propto iG$ because (2.2) shows that the phase of v differs by $\frac{1}{2}\pi$ from that of u and w.

On substituting (2.3) in (2.1) and (2.2) we obtain the following four scalar equations for the four unknown functions F, G, H, P:

$$\alpha(U-c)\,F+U'G = -\alpha P-i\nu\left\{F''+\frac{1}{r}F'-\left(\alpha^2+\frac{n^2}{r^2}\right)F\right\}, \tag{2.4}$$

$$\alpha(U-c)\,G = P'-i\nu\left\{G''+\frac{1}{r}G'-\left(\alpha^2+\frac{n^2+1}{r^2}\right)G-\frac{2n}{r^2}H\right\}, \tag{2.5}$$

$$\alpha(U-c)\,H = -\frac{n}{r}P-i\nu\left\{H''+\frac{1}{r}H'-\left(\alpha^2+\frac{n^2+1}{r^2}\right)H-\frac{2n}{r^2}G\right\}, \tag{2.6}$$

$$\alpha F+G'+\frac{1}{r}G+\frac{n}{r}H = 0 \tag{2.7}$$

† [*Note added in proof.*] An account of this work has now been published (Viilu 1962).

in which a prime denotes differentiation with respect to r. Equations containing only one of the unknown functions can be derived with no more than algebraic difficulty, but will not be needed here.

The boundary conditions (not necessarily independent) to be satisfied at the outer boundary are simply

$$F, G, H, P \to 0 \quad \text{as} \quad r \to \infty. \tag{2.8}$$

At the inner boundary, $r = 0$, the conditions to be satisfied are kinematic in origin. From the nature of the co-ordinate system both u_x and p must be independent of ϕ when $r = 0$, which requires, for $n \neq 0$,

$$F(0) = P(0) = 0. \tag{2.9}$$

The character of the co-ordinate system also requires both u_r and u_ϕ to vary as $\sin\phi$ in the neighbourhood of $r = 0$, which is compatible with (2.3) and non-zero values of $G(0)$ and $H(0)$ only if $n = 1$. Thus if $n \neq 1$ we must have

$$G(0) = H(0) = 0; \tag{2.10}$$

while if $n = 1$, non-zero values of $F(0)$ and $H(0)$ are possible and, in view of the form of the continuity relation (2.7),

$$G(0) = -H(0). \tag{2.11}$$

None of the above boundary conditions arises directly from the effect of viscosity, and are formally the same in an inviscid fluid.

Analysis of the stability of steady unidirectional flow in two dimensions has suggested that the net effect of viscosity is there such as to decrease the energy of a disturbance, except when the Reynolds number lies within certain ranges in cases of flow in the presence of a rigid boundary. In the absence of a rigid boundary, a disturbance of given wave-number to a given two-dimensional flow seems to have its greatest (non-dimensional) rate of growth at infinite Reynolds number, if instability to this type of disturbance exists at all; a given disturbance is then unstable at (not too small) finite Reynolds numbers only if it is unstable at infinite Reynolds number, although of course the effect of viscosity is relevant to the value of the Reynolds number below which no disturbances are amplified. The complexity of the equations (2.4) to (2.7) forces us to consider the stability of axisymmetric flows of inviscid fluid, in the first instance at any rate, and experience with two-dimensional flows encourages the belief that in the absence of a rigid boundary the results obtained allow firm deductions about whether a given disturbance is unstable at some finite Reynolds numbers.

The single governing equation for inviscid fluid

The governing equations (2.4) to (2.7) become, when $\nu = 0$, and after elimination of the pressure function P,

$$n(U-c)\,G + \frac{d}{dr}\{(U-c)\,rH\} = 0, \tag{2.12}$$

$$\alpha(U-c)\,(nF - \alpha rH) + nU'G = 0, \tag{2.13}$$

$$\alpha rF + rG' + G + nH = 0. \tag{2.14}$$

The first two of these equations are in effect equations for the x- and r-components of the disturbance vorticity respectively. Elimination of F between equations (2.13) and (2.14) gives

$$n(U-c)(rG'+G)-nrU'G+(n^2+\alpha^2r^2)(U-c)H = 0, \tag{2.15}$$

and then, on eliminating H between (2.12) and (2.15), we have the following equation in G alone:

$$(U-c)\frac{d}{dr}\left\{\frac{r}{n^2+\alpha^2r^2}\frac{d(rG)}{dr}\right\}-(U-c)G-rG\frac{d}{dr}\left(\frac{rU'}{n^2+\alpha^2r^2}\right) = 0. \tag{2.16}$$

The form of the solution near $r = 0$ may be seen immediately. For when $n \neq 0$, the leading terms in (2.16) give

$$\frac{d}{dr}\left\{r\frac{d(rG)}{dr}\right\}-n^2G = 0,$$

provided either $c_i \neq 0$ or $U' \neq 0$ at the point where $U = c_r$. This equation has solutions of the form r^{n-1} and r^{-n-1}, so that the boundary condition at $r = 0$ and the differential equation are both satisfied by

$$G(r) \sim r^{n-1} \tag{2.17}$$

near $r = 0$. In the case $n = 0$, the approximate form of (2.16) near $r = 0$ is

$$\frac{d}{dr}\left\{\frac{1}{r}\frac{d(rG)}{dr}\right\}-\alpha^2G = 0,$$

and the first terms of a series which satisfies this equation and the boundary condition (2.10) are

$$G(r) = C(r+\tfrac{1}{12}\alpha^2r^3), \tag{2.18}$$

where C is a constant.

Note on the use of the inviscid equation

As in the case of a primary flow which varies with respect to one lateral position co-ordinate, there is a singularity in the solution of the inviscid equation (2.16) at the critical point, $r = r_c$ at which $U = c$. If c is complex, then r_c is a point in the complex r-plane. In a fluid of small viscosity, this point is replaced by a critical layer $|r-r_c| < \delta$ whose thickness δ vanishes with the viscosity. For r_c real and non-zero, the critical layer is a thin cylindrical annulus in the fluid, and since the layer is thin, the local behaviour of the solution will be the same as in the case of a two-dimensional primary flow (as already remarked by Pretsch (1941) for the particular case of axisymmetric disturbances). For complex r_c the local behaviour is also the same as for two-dimensional primary flow. This means that, provided $r_c \neq 0$, the effect of friction in the critical layer of an axisymmetric jet can be obtained from known results for disturbances to a two-dimensional primary flow (for an account of which, see Tollmien 1929 and Lin 1955, chapter 8).

It is known that, as far as amplified disturbances are concerned, the effect of friction is negligible in the limit of vanishing viscosity, and that neutral disturbances also satisfy the inviscid equation, provided that when the logarithmic singularity appears at the critical point, the same branch of the logarithm is taken as for the slightly amplified case. For damped disturbances, the branch of the logarithm which must be taken corresponds to integrating (2.16) along a path in the complex r-plane which passes the critical point on the side *opposite* to the real axis (Lin 1955, pp. 125–6 and diagram p. 132). Consequently, only amplified and (if proper care is taken at the critical point) neutral solutions can be obtained by straight-forward integration of (2.16) along the real axis; for damped solutions, friction must be taken into account in certain regions (shown in Lin's diagram, p. 132) however small ν may be. The above remarks are of course subject to the proviso that $r_c \neq 0$, and the possibility of r_c being zero must be considered separately.

3. The complete solution for a cylindrical vortex sheet in inviscid fluid

For one particular jet-like profile it happens that the disturbance equations for an inviscid fluid can be solved completely. This profile is such that

$$U = U_0 \quad \text{for} \quad r < a$$

and

$$U = 0 \quad \text{for} \quad r > a,$$

and provides an approximate representation of the velocity distribution in a jet at positions close to a circular orifice through which fluid is emerging with uniform speed at high Reynolds number, as in an 'open-jet' wind tunnel. As well as having some direct interest, the results for this profile are useful as a means of testing the general theorems to be established later.

The feature of this case that makes it tractable is, of course, the irrotationality of the primary flow everywhere except on the cylinder $r = a$. Growing oscillations must likewise be irrotational except at the (displaced) vortex sheet, and it is more convenient to use this fact directly than to proceed via the governing equation (2.16). We suppose that the radial displacement of the vortex sheet due to the disturbance is $\eta(x, \phi, t)$, of which a normal mode is

$$\eta = A\, e^{in\phi + i\alpha(x-ct)}, \tag{3.1}$$

where A is a constant. The corresponding velocity potentials of the disturbance motion are

$$\left.\begin{matrix} \phi_0 = \Phi_0(r) \\ \phi_1 = \Phi_1(r) \end{matrix}\right\} \times e^{in\phi + i\alpha(x-ct)} \quad \text{for} \quad r \lessgtr a + \eta,$$

where Φ_0 and Φ_1 satisfy the modified Bessel equation

$$\Phi'' + \frac{1}{r}\Phi' - \left(\alpha^2 + \frac{n^2}{r^2}\right)\Phi = 0. \tag{3.2}$$

The general solution is of the form

$$\Phi(r) = CI_n(\alpha r) + DK_n(\alpha r), \tag{3.3}$$

where I_n and K_n stand, as usual, for the modified Bessel functions of the first and second kinds. For no value of n does $I_n(\alpha r) \to 0$ as $r \to \infty$; and for no value of n do K_n and K_n' vanish at $r = 0$ (nor, in the case $n = 1$, does $K_n' + iK_n$ vanish there). The boundary conditions (2.8), (2.9), (2.10) and (2.11) therefore require the solutions

$$\Phi_0(r) = CI_n(\alpha r), \quad \Phi_1(r) = DK_n(\alpha r). \tag{3.4}$$

This form for $\Phi_0(r)$ has a behaviour near the axis $r = 0$ which is consistent with the general relation (2.17) or, for $n = 0$, (2.18).

We now apply matching conditions at the common boundary of the two regions of irrotational motion. The kinematical condition that the boundary is a material surface gives

$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} = \left(\frac{\partial \phi_0}{\partial r}\right)_{r=a} \quad \text{and} \quad \frac{\partial \eta}{\partial t} = \left(\frac{\partial \phi_1}{\partial r}\right)_{r=a},$$

that is,

$$i\alpha(U_0 - c)A = \alpha C I_n'(\alpha a) \quad \text{and} \quad -i\alpha c A = \alpha D K_n'(\alpha a). \tag{3.5}$$

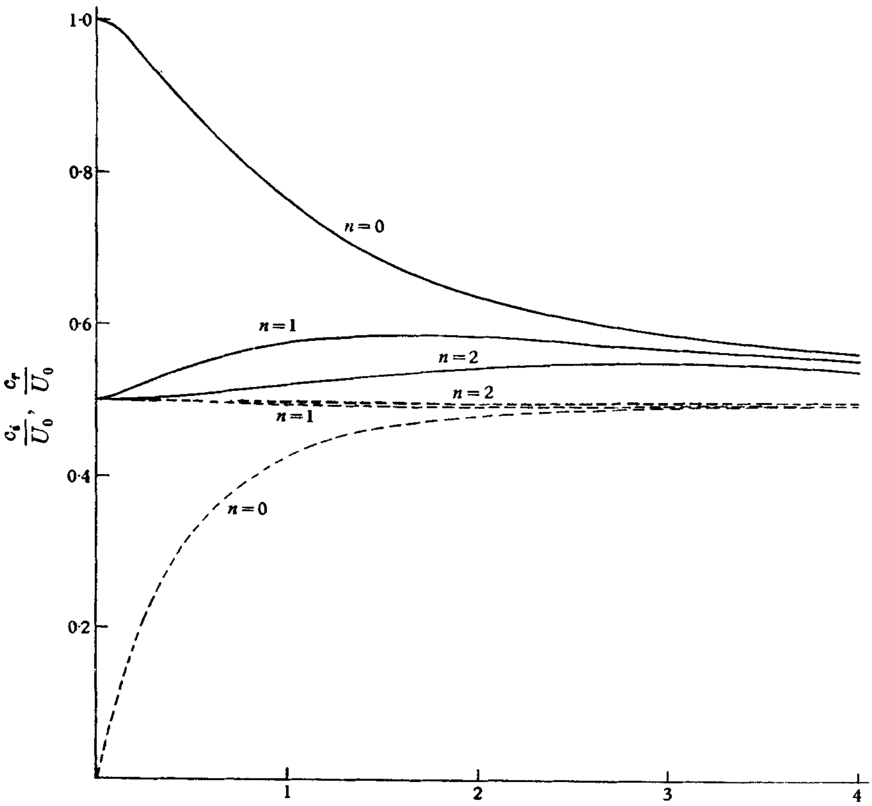

FIGURE 1. Wave speed c_r (solid lines) and amplification factor c_i (broken lines) for a disturbance to a cylindrical vortex sheet.

The dynamical condition that the pressure is continuous across the boundary gives

$$\left(\frac{\partial \phi_0}{\partial t} + U\frac{\partial \phi_0}{\partial x}\right)_{r=a} = \left(\frac{\partial \phi_1}{\partial t}\right)_{r=a},$$

that is,

$$(U_0 - c)\, C I_n(\alpha a) = -cDK_n(\alpha a). \tag{3.6}$$

Elimination of the constants A, C and D from the three relations (3.5) and (3.6) then gives, as the condition for a non-zero solution,

$$\left(\frac{U_0 - c}{c}\right)^2 = \frac{K_n(\alpha a)\, I_n'(\alpha a)}{K_n'(\alpha a)\, I_n(\alpha a)}, \quad = -L_n(\alpha a) \quad \text{say}, \tag{3.7}$$

where $L_n(\alpha a) \geqslant 0$ for all n and αa. The wave speed and amplification factor for a growing disturbance are then

$$c_r = \frac{U_0}{1+L_n(\alpha a)}, \quad c_i = \frac{L_n^{\frac{1}{2}}(\alpha a)\, U_0}{1+L_n(\alpha a)}. \tag{3.8}$$

The values of c_r/U_0 and c_i/U_0 as functions of αa for various values of n are shown in figure 1. The flow is unstable to a small disturbance for all values of n and αa. As $\alpha a \to \infty$, $L_n(\alpha a) \to 1$ for all n, and the results appropriate to a disturbance to a plane vortex sheet in two dimensions are recovered, as would be expected. At the other extreme $\alpha a \to 0$, corresponding to slow variation of the disturbance in the flow direction, we have $L_n(\alpha a) \to 1$, for $n \neq 0$ and $L_0(\alpha a) \to 0$. Thus long waves with axial symmetry travel with the speed of the centre of the jet, whereas all those not having axial symmetry travel with half that speed. The disturbance whose amplitude is increased by the largest multiplicative factor in a given time, with given values of U_0 and a, is the one for which αc_i is a maximum. This maximum occurs at the limit $\alpha \to \infty$, and is independent of n. However, it is to be expected that when the transition from the uniform velocity U_0 in the core of a jet to zero velocity outside the jet takes place in a layer of non-zero thickness, d say, disturbances for which $2\pi/\alpha \ll d$ are stable, just as in the case of a plane shearing layer. Provided $d \ll a$, the behaviour of disturbances to the cylindrical vortex layer for which αd is of order unity is approximately the same as for a plane layer, and we may expect that the maximum growth rate occurs when α is of order $1/d$.

4. General results for a unidirectional jet in inviscid fluid

Equation (2.16) for a disturbance to a unidirectional axisymmetric jet in inviscid fluid bears some resemblance to the equation for the stream function representing a (three-dimensional) disturbance to a plane unidirectional flow (and the latter equation may be recovered from (2.16) by putting $r = r_1 + y$, and letting r_1 tend to infinity while n/r_1 remains fixed), which suggests that it may be possible to find analogues of some of the well-known results for two-dimensional flow of inviscid fluid. This section describes a number of such analogous results.

A necessary condition for instability

A necessary condition for the existence of amplified disturbances can be obtained by dividing (2.16) by $(U-c)$, multiplying by the complex conjugate of rG, and subtracting the complex conjugate of the whole equation, to give

$$\frac{d}{dr}\left[\frac{r}{n^2+\alpha^2r^2}\left\{r\overset{*}{G}\frac{d(rG)}{dr} - rG\frac{d(r\overset{*}{G})}{dr}\right\}\right] = r^2\,|G|^2\,\frac{2ic_i}{|U-c|^2}\frac{d}{dr}\left(\frac{rU'}{n^2+\alpha^2r^2}\right). \tag{4.1}$$

The integral of the left-hand side over the range $r = 0$ to $r = \infty$ (or over any range such that $rG = 0$ at the two end points) is zero, and the consequence for the right-hand side is that

$$c_i\int_0^\infty |g|^2\, Q'\, dr = 0, \tag{4.2}$$

where $$g(r) = \frac{r}{U-c}G(r) \quad \text{and} \quad Q(r) = \frac{rU'}{n^2+\alpha^2r^2}. \tag{4.3}$$

Thus the quantity Q' must change sign at an interior point if amplified disturbances of the form (2.3) with $c_i \neq 0$ are to be possible. The parallel with the corresponding result for unidirectional flow in two dimensions is made plainer if we say that, for amplified disturbances to exist, U must have a point of inflexion with respect to the variable

$$\rho = \int \frac{n^2+\alpha^2 r^2}{r}\,dr = n^2 \log r + \tfrac{1}{2}\alpha^2 r^2. \tag{4.4}$$

The equation (2.16) and the necessary condition for instability that Q' changes sign were obtained by Rayleigh (1892). He pointed out that the velocity distribution in an exactly steady and unidirectional flow inside a rigid circular cylinder is of the form

$$U = A + Br^2 + C \log r,$$

with $C = 0$ unless a second concentric cylinder is present as an inner boundary to the fluid, and that the function

$$Q(r) = \frac{2Br^2 + C}{n^2 + \alpha^2 r^2}$$

varies monotonically with r. Thus amplified inviscid disturbances are not possible in this case. The particular case of an axisymmetric disturbance ($n = 0$) to Poiseuille flow in a circular tube ($C = 0$) is exceptional, in that Q' is then zero everywhere in the fluid, corresponding to a state of neutral stability like the two-dimensional case of plane Couette flow.

The implications of the above necessary condition for instability as applied to jets are especially illuminating. Near the centre of a jet we may put

$$U \approx U_0 - \beta r^2, \quad Q \approx -\frac{2\beta r^2}{n^2+\alpha^2 r^2} \quad (\beta > 0)$$

and, at the other end of the r-range, $Q \to 0$ from below as $r \to \infty$. There is evidently an important difference between the cases $n = 0$ and $n \neq 0$. For $n \neq 0$, $Q = 0$ at $r = 0$ and at $r = \infty$, so that there is certain to be a place in the fluid where $Q' = 0$, irrespective of the exact form of the velocity profile; the necessary condition for instability is satisfied. For $n = 0$, on the other hand, $Q = U'/\alpha^2 r$ and Q' may not change sign. For any slowly varying profile roughly like a Gaussian function, or for the profile (1.5) appropriate far downstream from a steady point-force, Q now increases monotonically from $r = 0$ (where $Q = -2\beta/\alpha^2$) to $r = \infty$, so that unstable oscillations are excluded; but for a profile which has a steeper gradient at some finite r than that for the parabola which fits the profile near $r = 0$ (so that $|U'| > 2\beta r$ at some r) Q does have a turning-point and unstable oscillations are not excluded (see figure 2). A 'top-hat' profile for which U is approximately constant in some central region and then falls rapidly to zero satisfies the necessary condition for instability to axisymmetric disturbances, and the instability has been found explicitly in § 3. Ring-shaped vortices at the boundary of the core are in fact a common feature of jets at positions close to an orifice at which the velocity is nearly uniform (Wehrmann & Wille 1958).

So far as profiles varying slowly with r are concerned, it seems that we must look to non-axisymmetric disturbances for an explanation of the observed instability of jets. It is shown later that at sufficiently large values of n axisymmetric jets are stable, and that for the particular profile (1.5) only sinuous modes, for which $n = 1$, are amplified.

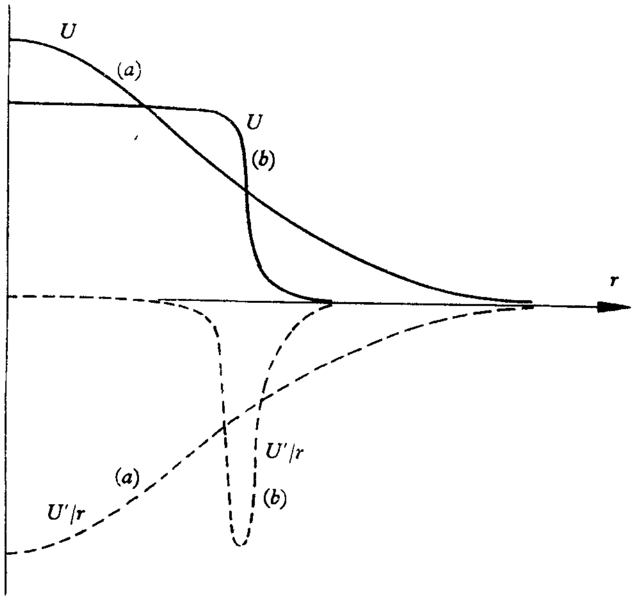

FIGURE 2. Jet profiles; (a) typical slowly varying profile, (b) 'top-hat' type of profile.

Refinement of the necessary condition

Fjørtoft (1950) and Høiland (1953) have shown that the necessary condition for instability of unidirectional inviscid flow in two dimensions, viz. that U'' changes sign somewhere in the fluid, can be made more specific and that, for profiles of moderately simple form, $|U'|$ must have a *maximum*. A similar refinement of the above necessary condition for instability of an axisymmetric jet can be made. Instead of subtracting in the last step leading to (4.1), we now add, and obtain

$$\frac{d}{dr}\left[\frac{r}{n^2+\alpha^2r^2}\left\{r\overset{*}{G}\frac{d(rG)}{dr}+rG\frac{d(r\overset{*}{G})}{dr}\right\}\right]$$
$$= \frac{2r}{n^2+\alpha^2r^2}\left|\frac{d(rG)}{dr}\right|^2+2r\,|G|^2+2r^2\,|G|^2\,Q'\,\frac{U-c_r}{|U-c|^2}. \tag{4.5}$$

Integration over the whole range of r then shows that

$$\int_0^\infty |g|^2\,(U-c_r)\,Q'\,dr < 0. \tag{4.6}$$

When $c_i \neq 0$, the integral in (4.2) is zero and we can replace c_r in (4.6) by any other constant factor, a convenient choice of which is the value of U at the point $r = r_f$ where $Q' = 0$, to be denoted by U_f; hence

$$\int_0^\infty |g|^2\,(U-U_f)\,Q'\,dr < 0. \tag{4.7}$$

Now both $U-U_f$ and Q' change sign together at $r = r_f$, in a case in which amplified disturbances are not excluded, and (4.7) shows that, in the case of profiles of such simple form that Q' changes sign only once, $U-U_f$ and Q' must have *opposite* signs. In other words, the point of inflexion that U must have with respect to the variable ρ defined in (4.4) must be one for which $|dU/d\rho|$ is a *maximum* with respect to ρ. In any monotonic jet profile, $U' \to 0$ from below, as $r \to \infty$, so that in the outer region of a jet $Q' > 0$; thus if Q' changes sign at all, it will do so in the manner required by (4.7).

Restrictions on c_r and c_i

The kind of juggling with the governing equation (2.16) that led to the relations (4.2) and (4.7) also yields integral relations which place restrictions on the wave speed c_r and the amplification factor c_i. We begin by noticing that (2.16) can be written in the form

$$\frac{d}{dr}\left\{\frac{r}{n^2+\alpha^2r^2}(U-c)^2 g'\right\} - \frac{1}{r}(U-c)^2 g = 0. \tag{4.8}$$

On multiplying by $\mathring{g}$ and integrating with respect to r, we find

$$\int_0^\infty (U-c)^2\,\Phi\,dr = 0, \tag{4.9}$$

where $$\Phi = \frac{r}{n^2+\alpha^2r^2}|g'|^2 + \frac{1}{r}|g|^2,$$

is a positive real function. The relation (4.9) cannot be satisfied if c is real and outside the range of U. (And if c is real and *inside* the range of U, the derivation of (4.9) fails because by (4.8) g and hence Φ may have a singularity at the point where $U = c$.) If $c_i > 0$, the imaginary and real parts of (4.9) yield, following Howard (1961),

$$\int_0^\infty U\Phi\,dr = c_r\int_0^\infty \Phi\,dr, \tag{4.10}$$

$$\begin{aligned}\int_0^\infty U^2\Phi\,dr &= \int_0^\infty (2c_rU - c_r^2 + c_i^2)\,\Phi\,dr \\ &= (c_r^2+c_i^2)\int_0^\infty \Phi\,dr \end{aligned} \tag{4.11}$$

by (4.10). Suppose now that $a \leqslant U(r) \leqslant b$. Then from (4.10) and (4.11) we have

$$\begin{aligned} 0 \geqslant \int_0^\infty (U-a)(U-b)\,\Phi\,dr &= \{c_r^2+c_i^2-(a+b)\,c_r+ab\}\int_0^\infty \Phi\,dr \\ &= \{[c_r-\tfrac{1}{2}(a+b)]^2+c_i^2-\tfrac{1}{4}(a-b)^2\}\int_0^\infty \Phi\,dr,\end{aligned}$$

i.e. $$\{c_r-\tfrac{1}{2}(a+b)\}^2+c_i^2 \leqslant \{\tfrac{1}{2}(a-b)\}^2. \tag{4.12}$$

Thus the complex wave velocity must lie within the semi-circle which has the range of U for diameter, exactly as for two-dimensional primary flows.

Components parallel and perpendicular to helices of constant phase

In considering a three-dimensional disturbance to a two-dimensional unidirectional primary flow, as in a proof of Squire's theorem, it is instructive to work in terms of components of the velocity parallel and perpendicular to the lines of intersection of the planes

$$\alpha x + \gamma z = \text{const.}, \quad y = \text{const.},$$

on which the phase of the disturbance wave is constant. An analogous procedure is useful here for several different purposes. For a disturbance of the form (2.3) to an axisymmetric jet, the lines of intersections of the surfaces

$$\alpha x + n\phi = \text{const.}, \quad r = \text{const.},$$

are circular helices on which the phase of the disturbance wave is constant. All these 'wave helices' advance an axial distance $2n\pi/\alpha$ in one complete turn, and the tangent to a helix at any point makes an angle $\tan^{-1}(\alpha r/n)$ with the direction of the axis. The components, in a cylindrical co-ordinate system, of the velocities of both the primary flow and the disturbance depend only on the variables r and ξ (for given α and n), where

$$\alpha\xi = \alpha x + n\phi, \tag{4.13}$$

and are constant on a helix of the above family. Consequently a material line defined at any instant by the relations $\xi = \text{const.}$, $r = \text{const.}$, continues to be a helix of constant phase, just as a material straight line of constant phase remains a straight line of constant phase in the case of two-dimensional primary flow.

The required new orthogonal components of the disturbance velocity $\mathbf{u}$ are

$$\left.\begin{aligned} u_1 &= \frac{nu_\phi}{rk} + \frac{\alpha u_x}{k}, && \text{perpendicular to both the radial line and the helix of constant phase,} \\ u_2 &= u_r, && \text{radial,} \\ u_3 &= \frac{\alpha u_\phi}{k} - \frac{nu_x}{rk}, && \text{parallel to the tangent to the local helix of constant phase,} \end{aligned}\right\} \tag{4.14}$$

where $k = (\alpha^2 + n^2 r^{-2})^{\frac{1}{2}}$ plays the part of a total wave-number magnitude. Now for a disturbance velocity which is a function of r and ξ alone, like the form (2.3), we may replace $\partial/\partial x$ by $\partial/\partial\xi$ and $\partial/\partial\phi$ by $(n/\alpha)\,\partial/\partial\xi$, so that the continuity equation (2.2) can be written as

$$\frac{k}{\alpha}\frac{\partial u_1}{\partial \xi} + \frac{1}{r}\frac{\partial(ru_2)}{\partial r} = 0.$$

This can be satisfied identically by introducing a stream function given by

$$u_1 = \frac{\alpha}{kr}\frac{\partial\psi}{\partial r}, \quad u_2 = -\frac{1}{r}\frac{\partial\psi}{\partial\xi}; \tag{4.15}$$

it is possible to show by the usual methods that $2\pi(\psi_A - \psi_B)$ is equal to the flux of fluid volume across the surface generated by one turn of the wave helices passing through any curve joining the two points A and B, and that ψ is constant on any one helix. The component u_3 parallel to the tangent to the local helix is analogous to the azimuthal component in a rotationally symmetric system.

Thus the disturbance is specified completely by the two scalar functions ψ and u_3, both of which are proportional to $e^{i\alpha\xi}$. The relation between ψ, u_3 and the functions F, G, H used earlier is evident from (2.3), (4.14) and (4.15):

$$\left.\begin{aligned} \psi(r,\xi) &= -\mathscr{R}\left\{\frac{r}{\alpha}G(r)\,e^{i\alpha(\xi-ct)}\right\},\\ u_3(r,\xi) &= \mathscr{R}\left\{\frac{(\alpha rH-nF)}{kr}\,e^{i\alpha(\xi-ct)}\right\}.\end{aligned}\right\} \tag{4.16}$$

The governing dynamical equation can be written with ψ as the dependent variable in place of G, and in addition we have from (2.13)

$$\hat{u}_3 = -\frac{n}{kr^2}\frac{U'}{U-c}\hat{\psi}, \tag{4.17}$$

where $\hat{u}_3$ represents the function of r such that

$$u_3 = \mathscr{R}\{\hat{u}_3\,e^{in\phi+i\alpha(x-ct)}\},$$

and likewise for $\hat{\psi}$. Equation (4.17) expresses the fact that the line integral of the total velocity (primary flow plus disturbance) round one turn of a material wave helix remains constant. $\hat{u}_3$ is known as soon as the eigen-function equation for G or $\hat{\psi}$ has been solved, and plays a passive part similar to that of the component of disturbance velocity parallel to the straight lines of constant phase in the corresponding problem of three-dimensional disturbances to a two-dimensional primary flow.

5. Properties of a neutral disturbance in inviscid fluid

The wave speed for neutral disturbances

The question of whether neutral disturbances exist has not yet been discussed, but if such disturbances are possible, we can show that the wave speed must have one of two values, by making use of (4.1). This equation may be written

$$\frac{dW}{dr} = -\frac{2c_i}{(U-c_r)^2+c_i^2}r^2\,|G|^2\,Q', \tag{5.1}$$

where
$$W = \frac{ir}{n^2+\alpha^2r^2}\{r\overset{*}{G}(rG)'-rG(r\overset{*}{G})'\} \tag{5.2}$$

and Q is defined by (4.3). We shall show later that W is related to a component of the Reynolds stress.

When c_i becomes small, (5.1) shows that the derivative of W also becomes small, except for a narrow peak near the point $r = r_c$, where $U = c_r$. So W itself will tend to a constant value on either side of $r = r_c$, but will have a jump at this point given by

$$\begin{aligned}[W] &= -\lim_{c_i\downarrow 0}\int_0^\infty \frac{2c_i}{(U-c_r)^2+c_i^2}r^2\,|G|^2\,Q'\,dr\\ &= -2\pi\left(\frac{r^2\,|G|^2\,Q'}{U'}\right)_{r=r_c},\end{aligned} \tag{5.3}$$

provided $U'_c \neq 0$. Now W must be zero both at $r = 0$ and $r = \infty$ in order to satisfy the boundary conditions, so for a monotonic profile the jump must be zero. If $r_c \neq 0$ (so that $U'_c \neq 0$) this requires that $Q' = 0$ at $r = r_c$, i.e. $r_c = r_f$, $c_r = U_f$. One would expect that the alternative $G = 0$ is not possible, and this can be verified, for example, by integrating (4.8) to obtain

$$(U-c)^2 g' = -\frac{(n^2+\alpha^2 r^2)}{r}\int_r^\infty \frac{(U-c)^2 g}{r}\,dr \quad (r > r_c), \tag{5.4}$$

and by noticing that if g is positive for large values of r, g increases as r decreases. Near $r = r_c$ (4.8) shows that g behaves like $(r-r_c)^{-1}$ or 1, but only the former case is consistent with (5.4). Therefore $G = (U-c)\,g/r$ tends to a non-zero value as $r \downarrow r_c$.

We have shown, then, that for neutral solutions which are the limit of amplified solutions, $c = U_f$, or possibly $c = U_0$.

Before looking into the latter possibility in more detail, it must be noted that the above deduction relies on being able to take the limit of amplified solutions, whereas these do not exist if Q' does not change sign within the fluid. At the same time, it would be surprising if (5.3) could not be deduced directly from the analysis for a neutral solution, and we will now demonstrate that (5.3) is valid for any neutral inviscid solution that is the limit of a viscous solution as the viscosity vanishes.

The examination of the solutions of (2.16) near the singularity $r = r_c$ ($r_c \neq 0$) shows that they are given by

$$rG_1 = (r-r_c)\,P_1(r-r_c), \tag{5.5}$$

$$rG_2 = P_0(r-r_c) + (Q'/Q)_{r=r_c}(r-r_c)\log(r-r_c)\,P_1(r-r_c), \tag{5.6}$$

where P_0, P_1 are power series with $P_0(0) = P_1(0) = 1$. The general solution is of the form

$$rG = ArG_1 + BrG_2.$$

G_2 is a multiple-valued function because of the logarithmic singularity, and this leads to the by now familiar problem of deciding which branch corresponds to the limit of a viscous solution. Tollmien (1929) solved this problem (which is discussed in detail by Lin 1955, chap. 8)), and concluded that the correct branch corresponds to a path in the complex plane above $r = r_c$ (for $U'_c < 0$). A little algebraic manipulation then reproduces (5.3), again under the restriction $r_c \neq 0$, so that our previous result is strengthened to 'any neutral disturbance has wave speed given by $c = U_f$ or $c = U_0$'.

Neutral disturbances with wave speed U_0

We can show that if neutral solutions with wave speed U_0 do exist, then they are singular in the sense that the values required by the boundary conditions at $r = 0$ are not approached as $r \to 0$. This may be possible because there is a singularity at $r = 0$, but to go into the question of existence fully the effect of viscosity near $r = 0$ must be taken into consideration.

The proof that the only solutions are singular ones is another application of (5.4). If g is positive for large values of r, (5.4) shows that g increases as r decreases. Near $r = 0$, (4.8) shows that if $r_c = 0$, g behaves like

$$r^{-2-\surd(n^2+4)} \quad \text{or} \quad r^{-2+\surd(n^2+4)} \quad \text{if} \quad n \neq 0,$$

or like

$$r^{-2} \quad \text{or} \quad 1 \quad \text{if} \quad n = 0,$$

but only the former alternative in each case is consistent with (5.4).

There is, however, one special case where the solution is non-singular, viz. when the total wave-number vanishes, i.e. $n = 0$, $\alpha = 0$. In this case the right-hand side of (5.4) vanishes, giving $g' = 0$ and

$$rG = U_0 - U(r), \tag{5.7}$$

which does not appear to satisfy the boundary condition at infinity. The reason for this can be seen by noting the form taken by the governing equation (2.16) as $\alpha \to 0$. (5.7) is a valid solution provided αr is small. For αr not small, (2.16) can be approximated by

$$\frac{d}{d(\alpha r)}\left\{\frac{d(\alpha r G)}{\alpha r d(\alpha r)}\right\} = G,$$

of which the solution matching (5.7) for r large is

$$rG = U_0 \alpha r K_1(\alpha r),$$

where K_1 is the modified Bessel function. Thus a uniformly valid approximation would be, for instance,

$$rG = U_0 \alpha r K_1(\alpha r) - U(r), \tag{5.8}$$

and this solution does satisfy the boundary condition. If the results for a cylindrical vortex sheet (§ 3) for large α are put in terms of rG, they take the form

$$rG = \begin{cases} 0 & (0 < r < a), \\ U_0 \alpha r K_1(\alpha r) & (r > a); \end{cases}$$

thus (5.8) is a uniformly valid approximation even for this discontinuous profile.

The energy equation and the Reynolds stress

The energy equation, averaged over x and ϕ, says that the rate of gain of energy of a cylindrical shell of fluid between $r = r_0$ and $r = r_0 + \delta r$ is equal to the flux of energy into the shell across the two cylindrical surfaces. The equation may be re-interpreted in the usual manner as a gain of energy under the action of pressure forces and Reynolds stresses on the cylindrical surfaces. The Reynolds stress on the cylindrical surface r = constant has two components, $-\rho\overline{u_1 u_2}$ perpendicular to the helices of constant flow, and $-\rho\overline{u_2 u_3}$ tangential to these helices, where u_1, u_2, u_3 are the components of $\mathbf{u}$ defined in (4.14).

By (4.15) and (4.16),

$$\overline{u_1 u_2} = \frac{i}{4kr}\{\dot{G}(rG)' - G(r\dot{G})'\}\, e^{2\alpha c_i t},$$

$$= \frac{k}{4r}\, W\, e^{2\alpha c_i t} \tag{5.9}$$

by (5.2). We have previously shown that $W \to 0$ as $c_i \downarrow 0$ if the wave speed is chosen so that $c_r = U_f$; thus the component of the Reynolds stress perpendicular to the helices of constant phase vanishes as the rate of amplification vanishes. Also, if $c_r = U_0$, this component of the Reynolds stress will vanish everywhere, except possibly near $r = 0$.

For the component of the Reynolds stress parallel to the local helix of constant phase, we have, by (4.15), (4.16) and (4.17),

$$\overline{u_2 u_3} = \frac{1}{4ik}\left\{\frac{nU'}{\alpha r(U-c)}G\overset{*}{G} - \frac{nU'}{\alpha r(U-\overset{*}{c})}\overset{*}{G}G\right\}e^{2\alpha c_i t}$$

$$= \frac{nU'c_i}{2\alpha\{(U-c_r)^2+c_i^2\}}\frac{|G|^2}{rk}e^{2\alpha c_i t}, \tag{5.10}$$

$$\to \frac{2\pi n}{\alpha}\left(\frac{|G|^2}{rk}\right)_{r=r_c}\delta(r-r_c) \quad \text{as} \quad c_i \downarrow 0 \quad (r_c \neq 0). \tag{5.11}$$

Thus, in the limit of zero amplification rate $\overline{u_2 u_3}$ vanishes except for a peak near $r = r_c$. This peak can be important because any integral of this Reynolds stress over a region containing the critical point $r = r_c$ will remain finite as $c_i \downarrow 0$. For instance, the rate at which the Reynolds stresses do work over the whole fluid (per unit axial length of the jet) is

$$-\pi\rho\int_0^\infty U'\,\overline{u_x u_r}\,r\,dr = -\pi\rho\int_0^\infty U'\left(\frac{\alpha}{k}\overline{u_1 u_2} + \frac{n}{rk}\overline{u_2 u_3}\right)r\,dr$$

$$\to \frac{2\pi^2 n^2}{\alpha}\rho\left(\frac{-U'G^2}{rk^2}\right)_{r=r_c} \quad \text{as} \quad c_i \downarrow 0 \tag{5.12}$$

by (5.11). This rate of doing work is presumably equal to the rate of gain of disturbance energy, but it will be noticed that it does not vanish as the amplification rate vanishes! This unexpected result associated with the tangential Reynolds stress presumably implies that viscous dissipation in the neighbourhood of the critical point absorbs the excess production of energy however small ν may be.

A singularity in the tangential component of disturbance velocity

Looking back at the derivation of (5.10) and (5.11) we see that this peak in the Reynolds stress is associated with a singularity in the component of disturbance velocity tangential to the helices of constant phase. By (4.15), (4.16) and (4.17),

$$\hat{u}_3 = -\frac{n}{(n^2+\alpha^2 r^2)^{\frac{1}{2}}}\frac{U'}{\alpha(U-c)}i\hat{u}_2, \tag{5.13}$$

showing that this tangential component becomes infinite at the critical point $r = r_c$ where $U = c$. This singularity also appears in the case of unidirectional flows in two dimensions (Benney 1961, pp. 221 *et seq.*), where the corresponding equation (in the usual notation) for the component of disturbance velocity parallel to the line of constant phase of a three-dimensional disturbance is

$$\hat{u}_3 = -\frac{\gamma}{(\alpha^2+\gamma^2)^{\frac{1}{2}}}\frac{dU/dy}{\alpha(U-c)}i\hat{u}_2,$$

which may be recovered from (5.13) under the limiting process $r = r_1 + y, \gamma = n/r_1$, $r_1 \to \infty$. Physically (4.17) shows that the momentum per unit volume of the basic flow has a component $\rho(n/rk)\,U$ tangential to the wave helix which is convected radially by the disturbance component u_2. This gives rise to a tangential component of disturbance momentum, which is carried away by the mainstream except at the point where $U = c$.

Hence if an initial value problem were under consideration, the amplitude of a disturbance would grow with time near the critical point. For a real fluid, viscosity would become more important as the velocity gradient became larger, and eventually an equilibrium would be reached, with viscous diffusion just balancing the influx of momentum by convection. Thus for a real fluid the singularity at the critical point would be replaced by a narrow 'critical layer' in which viscosity is important. To find the behaviour of the tangential velocity component in the critical layer, the viscous terms in the equation (2.4) to (2.6) would have to be taken into consideration. As in the two-dimensional case, the critical layer has thickness $(\nu/\alpha)^{\frac{1}{3}}$ and an appropriate new space variable is $s = (r-r_c)(\nu/\alpha)^{-\frac{1}{3}}$. By (4.17)

$$\hat{u}_3 \sim -\frac{n}{r_c^2(r-r_c)}\left(\frac{\hat{\psi}}{k}\right)_{r=r_c} = -\left(\frac{\nu}{\alpha}\right)^{-\frac{1}{3}}\frac{n}{r_c^2 s}\left(\frac{\hat{\psi}}{k}\right)_{r=r_c}$$

as the critical layer is approached, so $\hat{u}_3$ should also be scaled as

$$\hat{u}_3 = (\nu/\alpha)^{-\frac{1}{3}}\,\omega(s).$$

The details of the behaviour of ω as a function of s in the critical layer are the same as for the two-dimensional case. The equation satisfied by $\omega(s)$ can be obtained by making the relevant approximations to (2.4) and (2.6) and then subtracting n/r times (2.4) from α times (2.6). In terms of ω and s it is

$$i\frac{d^2\omega}{ds^2} + U_c' s\omega = -\frac{n}{r_c^2}\,U_c'\left(\frac{\hat{\psi}}{k}\right)_{r=r_c},$$

and the solution can be expressed in terms of a Lommel function. (Compare $\omega(s)$ with the function $\hat{\phi}$ in the paper by Benney 1961, p. 222.)

If some disturbances are amplified exponentially according to the inviscid theory, the effect of the singularity in the velocity component parallel to the local wave helix will not be important, but in other cases, such as for the boundary layer formed on a cylinder moving parallel to its generators, this singularity can be expected to play an important role when non-linear interactions are considered.

A necessary condition for the existence of a neutral disturbance with $c_r = U_f$ $(n \neq 0)$

Unfortunately the type of argument used to show the existence of neutral disturbances for unidirectional flows in two dimensions is more successful here in showing when solutions do *not* exist rather than when they do. In fact it can be shown that neutral disturbances with $c = U_f$ do not exist when n is too large.

If $\beta = \alpha/n$, (2.16) may be written as

$$r\frac{d}{dr}\left\{\frac{r}{1+\beta^2 r^2}\frac{d(rG)}{dr}\right\} = \{n^2 - N^2(r,\beta)\}\, rG, \tag{5.14}$$

where N^2 is the positive valued function

$$N^2(r,\beta) = \frac{r}{U_f - U(r)}\frac{d}{dr}\left(\frac{rU'}{1+\beta^2 r^2}\right). \tag{5.15}$$

When $n < N$, the solutions of (5.14) are oscillatory in character, whereas when $n > N$ they are monotonic. If n is greater than the largest value of N, the solution will be everywhere monotonic, and the boundary conditions ($rG = 0$ both at $r = 0$ and at $r = \infty$) cannot be satisfied. Thus a necessary condition for the existence of a solution is

$$n^2 < \max_{r,\,\beta} N^2(r,\beta). \tag{5.16}$$

This condition is quite restrictive. For the jet profile (1.5) it is shown in the next section that no solution exists for $n \geqslant 2$, thus leaving $n = 1$ as the only possibility for a non-singular solution.

6. Demonstration that there is one, and only one, neutral disturbance with $\alpha \neq 0$ for the jet profile (1.5)

In the case of an experimental jet produced by discharge of fluid from an orifice (see § 1), the velocity in the laminar jet is given by (1.5), except near the orifice, if the Reynolds number R is large. If the condition (1.11) is satisfied, r_0 and U_0 can be treated as constants, so let us choose our length and time scales in such a way that both r_0 and U_0 are unity. Then (1.5) becomes

$$U = (1+r^2)^{-2}. \tag{6.1}$$

For amplified disturbances to exist, Q' must change sign in the fluid. For the profile (6.1) we have for Q (see the definition (4.3))

$$Q(r) = \frac{-4r^2}{(1+r^2)^3\,(n^2+\alpha^2 r^2)},$$

and $Q' = 0$ at the critical point $r = r_c$ given by

$$3\left(\frac{\alpha r_c}{n}\right)^4 + 2\left(\frac{\alpha r_c}{n}\right)^2 - \left(\frac{\alpha}{n}\right)^2 = 0 \tag{6.2}$$

for $n \neq 0$. It is convenient to write $(\alpha r_c/n)^2 = q$, and $\alpha/n = \beta$ as before, in which case (6.2) can be written as

$$3q^2 + 2q = \beta^2. \tag{6.3}$$

For $n = 0$, Q' does not change sign in the fluid, and there are no amplified solutions.

We shall use the governing equation in the form (5.14). The quantity N^2 is found from (5.15), (6.1) and (6.3) to be

$$N^2 = \frac{72r^2(1+3qr^2)\,(1+q)^2}{(1+r^2)^2\{1+(2+3q)\,qr^2\}^2\{(5+6q)+(2+3q)\,r^2\}}. \tag{6.4}$$

A graph of the function N^2 as a function of r^2 for various values of q is reproduced in figure 3. The maximum value of N^2 occurs for a very small value of q and is approximately 2·73. The necessary condition (5.16) for the existence of a neutral solution with $c = U_f$ shows that in this case neutral solutions do not exist for $n \geqslant 1$. It is commonly found that two-dimensional flows are stable at large values of the wave-number magnitude; and presumably the total wave-number is so large here when $n > 1$ that disturbances are stable.

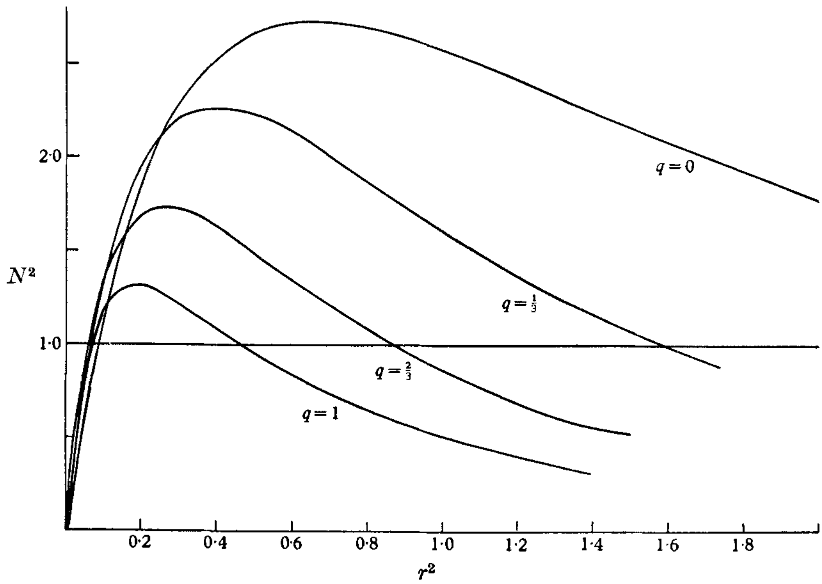

FIGURE 3. N^2 as a function of r^2 for various values of q.

For $n = 1$, it remains to show that there exists a neutral solution of finite wave number α_0. This can be accomplished by considering the solutions of (5.14) which satisfy the boundary condition $rG = 0$ at $r = 0$ for extreme values of α (i.e. extreme values of q) and assuming that the solutions vary continuously for intermediate values. First, for q large, (6.4) shows that $N^2 \to 0$ so that (5.14) becomes approximately

$$r\frac{d}{dr}\left\{\frac{r}{1+\alpha^2r^2}\frac{d(rG)}{dr}\right\} = rG,$$

of which the solution (for rG) which vanishes at $r = 0$ is

$$rG = rI_1'(\alpha r), \tag{6.5}$$

which is everywhere positive. On the other hand, for $q \to 0$, (5.14) can be approximated by

$$r\frac{d}{dr}\left\{r\frac{d(rG)}{dr}\right\} = \left\{1-\frac{72r^2}{(1+r^2)^2(5+2r^2)}\right\}rG, \tag{6.6}$$

except for r very large ($\alpha r = O(1)$), where the solution of this equation must be matched to a Bessel function. Equation (6.6) has a solution which is an algebraic function, viz.

$$G = \frac{(5+2r^2)(5-r^2)}{(1+r^2)^2}.$$

(This solution was first obtained by removing the singularity at $r^2 = -1$ from the equation, and then expanding $(1+r^2)^2G$ as a power series.) The important feature of this solution is that G changes sign, whereas when q is large G does not change sign but goes rapidly to ∞. Presumably for some finite value of q, q_0 say, G just fails to change sign, and satisfies the boundary conditions at both $r = 0$ and $r = \infty$.

The Cambridge computer EDSAC was used to find q_0. The programme was designed to integrate a modified form of (5.14) by the Runge–Kutta–Gill procedure, starting at ∞. In this way the behaviour near $r = 0$ of solutions of (5.14) satisfying the boundary condition at $r = \infty$ was obtained for various values of q, and an interpolation procedure was used to find the value q_0 for which the boundary condition at $r = 0$ was also satisfied. The value obtained was $q_0 = 0{\cdot}57$, corresponding to a wave-number $\alpha_0 = 1{\cdot}46$.

7. Conclusion

So far as inviscid disturbances are concerned, the jet profile (1.5) (and, it may be supposed, other profiles roughly like it) is unstable only when $n = 1$ (corresponding to a 'sinuous' type of disturbance) and the axial wavelength is larger than some critical value several times the jet diameter. The next step in the whole theoretical problem is to examine the effect of viscosity and to determine the critical Reynolds number of the jet. Viscosity appears to have a purely stabilizing effect in cases of two-dimensional unidirectional primary flows in the absence of rigid boundaries, and it would be reasonable to base further analysis of jet stability on the assumption that the same is true of axisymmetric primary flows. Thus analysis of the stability of a jet with a profile like (1.5) at finite Reynolds number can assume at the beginning that $n = 1$, if that helps. The observations of a jet reported by Reynolds (1962) do suggest that under certain conditions a sinuous type of disturbance of large wavelength is amplified.†

It is very important to know if the critical Reynolds number is large. If it is, a neutral disturbance exists only under conditions such that (*a*) the approximation of unidirectionality of the primary flow streamlines is accurate, and (*b*) the conventional approximations of stability theory associated with high Reynolds numbers may be made; if it is not, jets are stable only when the streamlines of the primary flow are inclined to each at appreciable angles and new methods of analysing the stability will be needed. Again the observational evidence is helpful; Reynolds (1962) observed, as also did Schade, that a jet from a small orifice can be steady over an axial distance large compared with the average jet diameter at Reynolds numbers larger than 100. It would therefore be sensible to assume, in a preliminary theoretical determination of the critical Reynolds number of a jet with a profile like (1.5), that the Reynolds number is large at all points on the curve of neutral stability in the (R, α)-plane; the validity of the assumption would of course be revealed by the outcome of the analysis.

† The correspondence between observations of growing disturbances and the theory of small disturbances to unidirectional flow is complicated by the fact that the diameter of a real jet increases with distance downstream, thereby changing the effective non-dimensional wave-number of a disturbance of given absolute frequency, and is beset by difficulties not discussed here.

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